Mathematics of a coilgun

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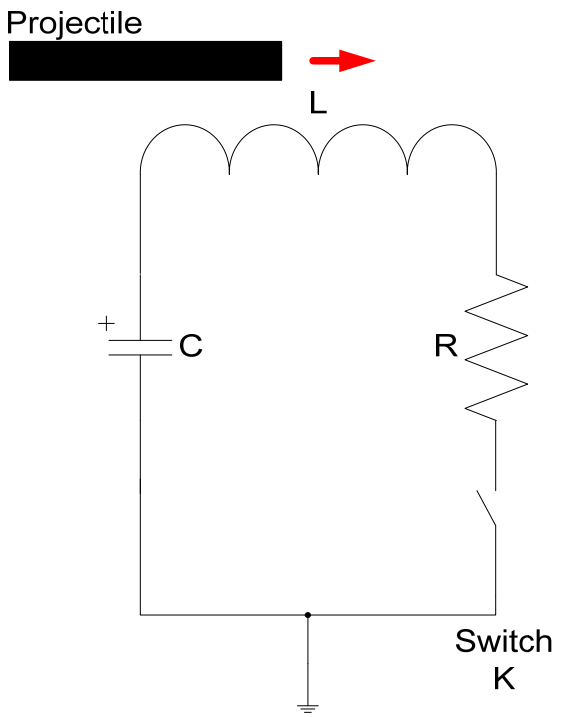
1. Introduction

Basic equations for current and voltage in *RLC*-circuit of one-stage coilgun are presented in this article. Using them, some interesting specific cases of practical importance are analyzed (for example, current decay in a half-bridge recuperative coilgun). To simplify the perception all formulae are cited in an explicit (non-complex) form.

2. Basic equations

Let us suggest a simple *RLC*-circuit (fig.1). Assuming applied as a coilgun, it consists of an accelerating coil inductance L, active resistance R (of a coil wire, capacitor, switch K and interconnections) and capacitance C. Practice demonstrates that other parasitic elements of a circuit (like interwinding capacitance and internal inductance of a cap) can be neglected. Typical values of R, L and C for amateur coilguns are in the limits shown in table 1.



Parameter	Dimension	Range
Active resistance, *R*	Ohm	0.1 … 5
Coil inductance, *L*	µHn	10 … 1000
Capacitance, *C*	µF	10 … 10 000
Initial voltage, $U_C(0)$	V	50 … 1000

Table 1. Typical range of parameters for the amateur coilguns.

Fig. 1. Typical coilgun circuit.

To find current and voltage dependences *i(t)* and *Uc(t)* one should write equations for voltages in circuit (the second Kirchhoff law):

$$U_C(t) = U_L(t) + U_R(t) \qquad (1)$$

where $U_L(t) = L \cdot di/dt$ is voltage in a coil (assuming inductance is constant), $U_R(t) = i(t) \cdot R$ is ohmic voltage drop.

then, writing capacitance through the charge

$$C = q(t)/ U_C(t) \qquad (2)$$

and differentiating (1) on *t*, one has a final equation for the charge in capacitor:

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0 \qquad (3)$$

which is a basis for description of any process in the circuit. For example, dividing the solution on *C*, we according to (2) have the voltage $U_C(t)$, and differentiating it on *t* gives current in the circuit *i(t)*.

(3) is second-order differential equation. Its solution depends on initial conditions. Assuming a coilgun circuit after the switches are on, one should write:

$$\left.\begin{aligned} i(0) &= 0 \\ q(0) &= Uc(0) \times C \end{aligned}\right\} \qquad (3a)$$

where $U_C(0) = U_0$ – initial capacitor voltage.

Beginning to solve (3) with (3a), we find out that solution splits to 2 cases depending on the value $k = 4L/R^2C$. Lets write them for current and voltage, at first for $k > 1$:

$$U_c(t) = U_0 e^{-\frac{tR}{2L}} \left[\cos\left(\frac{tR}{2L}\sqrt{\frac{4L}{R^2C} - 1} \right) + \frac{1}{\sqrt{\frac{4L}{R^2C} - 1}} \sin\left(\frac{tR}{2L}\sqrt{\frac{4L}{R^2C} - 1} \right) \right]$$

$$i(t) = \frac{2U_0}{R} \frac{e^{-\frac{tR}{2L}}}{\sqrt{\frac{4L}{R^2C} - 1}} \sin\left(\frac{tR}{2L}\sqrt{\frac{4L}{R^2C} - 1} \right) \qquad (4a)$$

For $k < 1$:

$$U_c(t) = U_0 e^{-\frac{tR}{2L}} \left[ch\left(\frac{tR}{2L}\sqrt{1 - \frac{4L}{R^2C}} \right) + \frac{1}{\sqrt{\frac{4L}{R^2C} - 1}} sh\left(\frac{tR}{2L}\sqrt{1 - \frac{4L}{R^2C}} \right) \right]$$

$$i(t) = \frac{2U_0}{R} \frac{e^{-\frac{tR}{2L}}}{\sqrt{1 - \frac{4L}{R^2C}}} sh\left(\frac{tR}{2L}\sqrt{1 - \frac{4L}{R^2C}} \right) \qquad (4b)$$

Where $sh(x) = \frac{e^x - e^{-x}}{2}$ is hyperbolic sine, $ch(x) = \frac{e^x + e^{-x}}{2}$ is hyperbolic cosine.

The direction of current leading to discharge of the capacitor is assumed here as positive.

One should note that (4a) and (4b) look similar with only difference in substitution of (1-k) to (k-1), and hyperbolic sine and cosine - to trigonometric ones. Looking ahead, it should be remarked that this analogy is true for all other calculations (for example, equations for power etc). This allows us to find a solution for the only case and expand it to another one by simply modifying all functions from hyperbolic to trigonometric.

The functions (4) are depicted in fig. 2.

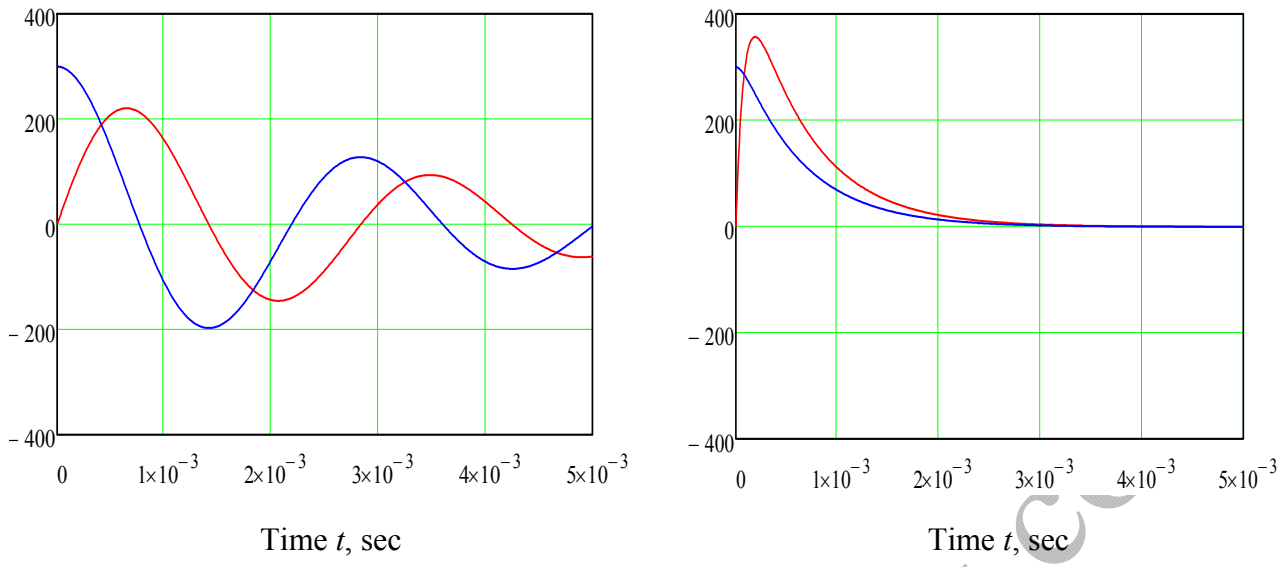


Fig. 2. Capacitor voltage (—) and current (—) depending on time for underdamped (left) circuit (*R* = 0,3 Ohm, *C* = 400 mkF, *L* = 500 mkHn) and overdamped (right) one (*R* = 0,7 Ohm, *C* = 1000 mkF, *L* = 50 mkHn). Initial capacitor voltage 300 V.

It is seen that (4a) corresponds to an oscillating discharge (i.e. periodic process takes place accompanied with change of the sign of the current and voltage across the capacitor). Such a circuit (with $k<1$) is called "underdamped".

(4b) presents so-called "overdamped" circuit with monotonically decaying voltage on capacitor.

Trying to simplify (4), one can introduce the values $\tau_L = 2L/R$ ("inductive constant" of the circuit, inverse to "decay constant" in the "classic" circuit theory) and "capacitive constant" $\tau_C = RC/2$. Then the solutions for periodic and non-periodic discharge will be:

$$U_c(t) = U_0 e^{-\frac{t}{\tau_L}}\left[ch\left(\frac{t}{\tau_L}\sqrt{1-k}\right) + \frac{1}{\sqrt{1-k}}sh\left(\frac{t}{\tau_L}\sqrt{1-k}\right)\right]$$

$$i(t) = \frac{2U_0}{R}\frac{e^{-\frac{t}{\tau_L}}}{\sqrt{1-k}}sh\left(\frac{t}{\tau_L}\sqrt{1-k}\right) \qquad (5a)$$

$$U_c(t) = U_0 e^{-\frac{t}{\tau_L}}\left[\cos\left(\frac{t}{\tau_L}\sqrt{k-1}\right) + \frac{1}{\sqrt{k-1}}\sin\left(\frac{t}{\tau_L}\sqrt{k-1}\right)\right]$$

$$i(t) = \frac{2U_0}{R}\frac{e^{-\frac{t}{\tau_L}}}{\sqrt{k-1}}\sin\left(\frac{t}{\tau_L}\sqrt{k-1}\right) \qquad (5b)$$

The condition of transition from the former solution to the latter one is equality to 1 of the previously mentioned parameter $k = \tau_L/\tau_C$. This parameter is hereinafter called "damping coefficient" or simply "k – parameter". It varies in a substantial range for amateur coilguns (as shown in table 1), so either overdamped or underdamped circuits can take place.

Thus, we have found out the solutions of the main equations for a coilgun circuit. All subsequent investigation is based on their analysis in some particular cases.

Notes:

1) The negative voltage values during periodic discharge mean that the underdamped regime cannot be used in circuits with electrolytic capacitors. Most probably there will not be any visible consequences for a real circuit in such a situation (I used underdamped circuit in my early experiments, and Evgeny Vasiliev has even utilized a small electrolytic cap recharging to high negative potential to quicken the current decay in his V-switch scheme). However, manufacturers of the electrolytic capacitors do not recommend them to be used in this mode in power circuits. The solution is to shunt a cap with reverse-biased diode – this limits the negative voltage across the cap with 0.5…1 V which is permissible for aluminum capacitors. But one should keep in mind that this diode leads to prolongation of the current decay from the curves described by (5) to more simple but much longer exponential form $i(t) \sim exp(-t \cdot R/L)$.

2) Definitions of the inductive and capacitive constants are not accidental. First, they substantially simplify the formulae and all subsequent calculations. Second, they absorb and divide all constants relative to inductance and capacitance and make all equations intuitively clear. For example, overdamping condition is written as $k < 1$ and means that the capacitance dominates in a circuit and doesn't allow current to change its direction, or the active resistance is so large that the current decays too fast to become periodic.

3) One should note that all presented equations describe an ideal *RLC*-circuit which can significantly differ from a real one. For instance, the coil inductance is changing during a shot as a ferromagnetic projectile moving in a barrel, and the voltage across a coil is really $U_L(t) = L(t) \cdot \{di(t)/dt\} + i(t) \cdot \{dL(t)/dt\}$ (I used this property in my original inductive position sensor). Besides, ohmic heat produced in wires varies their resistance, and capacitance can also change because of some effects like dielectric absorption. Thus, none of the circuit parameters can be considered constant. However, some investigations show that these variations are no more than few percents from the initial values, and can be neglected for accessing calculations.

4) From the mathematical point of view so called "critical decay", when $k = 1$, is of interest. In this case the equations (5) reduce to a simpler form:

$$U_c(t) = U_0 \cdot \left(1 + \frac{t}{\tau_L}\right) \cdot e^{-\frac{t}{\tau_L}}$$

$$i(t) = \frac{2U_0}{R} \frac{t}{\tau_L} e^{-\frac{t}{\tau_L}} \qquad (5c)$$

The feature of this condition is the fastest decay of current and voltage, however not becoming periodic.

3. Zeroes and maximums

Let us examine the behavior of the coilgun circuit basing on the previously found equations.

For the beginning, we can find the maximum value of current and the moment of time when it realizes. This is a problem of high practical importance because its solution allows an adequate choice of power switches – the "weak" switches will burn out, and too "strong" ones will be too expensive.

Equaling the time derivative of (5a, b) to zero, we have:

$$t_{\max} = \frac{\tau_L}{\sqrt{k-1}} arctg(\sqrt{k-1}) \qquad (6a)$$

(for the periodic discharge the first zero is taken as it corresponds to the maximal current)

$$t_{\max} = \frac{\tau_L}{\sqrt{k-1}} arth(\sqrt{1-k}) \qquad (6b)$$

Inserting these values to (5) we can calculate the currents and voltages at the corresponding moments:

$$i_{\max} = i(t_{\max}) = \frac{2U_0}{R} \frac{e^{-\frac{arctg\sqrt{k-1}}{\sqrt{k-1}}}}{\sqrt{k-1}} \sin(arctg\sqrt{k-1})$$

$$U_c(t_{\max}) = U_0 \frac{2e^{-\frac{arctg\sqrt{k-1}}{\sqrt{k-1}}}}{\sqrt{k-1}} \sin(arctg\sqrt{k-1}) \qquad (7)$$

(for an overdamped circuit all equation will be the same but changing trigonometric functions to hyperbolic ones).

Note that $U_C(t_{max}) = I(t_{max})\cdot R$, i.e. the voltage and current at t_{max} are ruled by Ohm's law accounting only active resistance. It's interesting to analyze the difference between the maximum current in coilgun and corresponding value in simple *RC*-circuit $I_m = U_0/R$ (without the coil). To do this the value i_{max}/I_m as a function of k is depicted in fig. 3.

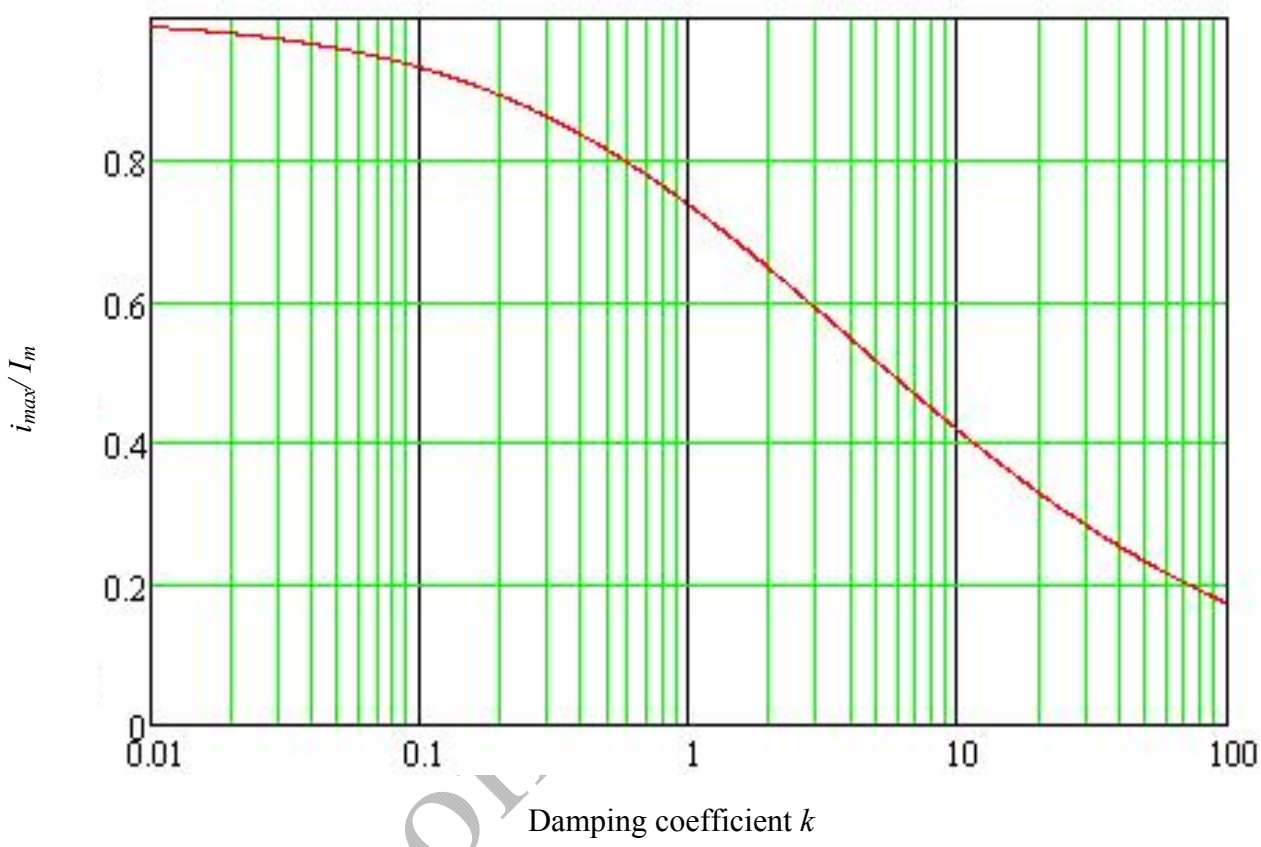


Fig. 3. Ratio of maximum current i_{max}, reached in RLC-circuit with initial cap voltage U_0, to current $I_m = U_0/R$ of inductorless RC-circuit.

As it is seen, for strongly overdamped systems i_{max} is very close to I_m, for critical decay $i_{max}/I_m \approx 0.73$, and i_{max}/I_m can be much smaller for strongly underdamped systems. But k is rarely more than ≈ 30 in real circuits, so one can say that assessment $i_{max} \approx U_0/R$ gives no more than 2-3 times overestimation.

It is useful to estimate positive current semi-wave duration for an underdamped circuit. Equaling (5) to zero and neglecting $t = 0$ solution, we get:

$$t_0 = \frac{\pi\tau_L}{\sqrt{k-1}} \qquad (8)$$

Voltage across the cap at this moment is

$$U(t_0) = -U_0 e^{-\frac{\pi}{\sqrt{k-1}}} \qquad (9)$$

Note that the voltage is negative. When $k \to \infty$ (i.e. for strongly underdamped circuits) $U(t_0) \to -U_0$, and $t_0 = \pi\sqrt{\tau_L \tau_C}$ - well known Thomson's formula for the oscillation period in *LC*-circuit.

Notes:

1) Maximum pulsed current for the switches chosen in a given circuit must be not less than value of (7). Besides, the chips' temperature can be estimated through so called "thermal resistance" Z_{th}, which connects the power dissipation P and temperature rise ΔT by the following equation $\Delta T = Z_{th} \cdot P$. As the impulse process is assumed, the dynamic value (i.e. depending on the pulse duration τ) $Z_{th}(\tau)$ must be used instead of the static one (it is given in power semiconductor's datasheets). For rough estimations one may use $\tau = t_{max}$ given by (6), and power $P = i_{max} \cdot u$, where u is voltage drop across the switch at current i_{max} (may be found in an according datasheet). In reality the power is of course function of time, but in this method it is approximated by constant value. The final chip temperature (accounting for its initial value and ΔT assessed by the formulae above) being no more than permissible limits, the switch can be used in a coilgun.

2) The closeable switch is being used, the formulae for currents and voltages can be applied to estimate a value of damping resistance R_d (see here). If the switch is closed on the moment t, the voltage rise will be $R_d \cdot i(t)$, and summary voltage across the switch will be $Uc(t) + R_d \cdot i(t)$. The latter value must be no more than maximum permissible voltage for the switch (better, about 30% less).

Imagine that the scheme faults, or the time t differs from the preliminarily calculated. In this case the voltage spike may be more than suspected. So, when estimating $i(t)$ and R_d , one should better use maximal value i_{max}. It guarantees the switch from damaging at any pulse durations (i.e. the circuit will be "fool-protected").

3) The calculations in FEMM show that the maximum acceleration efficiency is reached when the current in "zero point" (i.e. when a projectile is in center of a coil and the magnetic force becomes braking and "suck-back" effect begins) is about 30% of its maximal value. So, taking *RLC*-parameters of the given coilgun, and current pulse duration, one can use (5a) and (5b) formulae to estimate the coilgun's efficiency.

4. Equations for non-zero initial current in a coil

Now let's try to solve the basic equation for *RLC*-circuit for the case when the current i_0 is already running when the switch K opens (assume i_0 is charging the capacitor, see fig. 4).

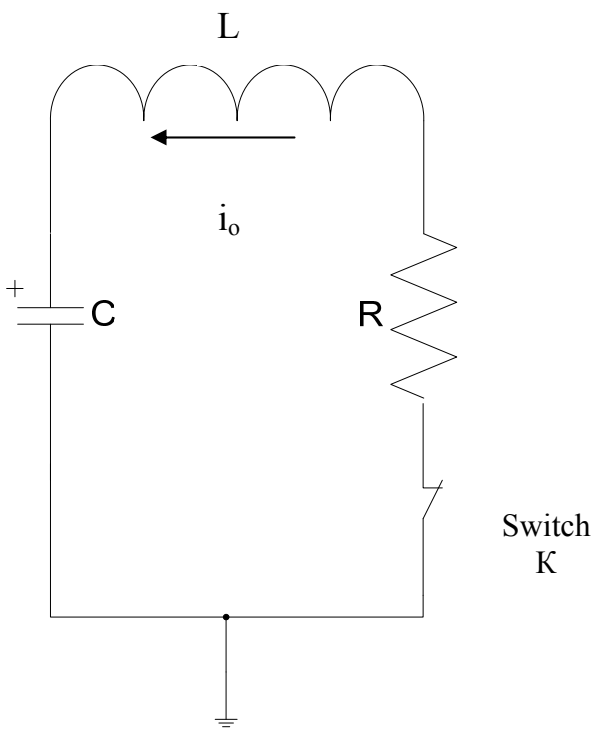


Fig.4. Illustration for the case of non-zero initial current in *RLC*-circuit.

This case describes the demagnetization cycle of flyback converters widely used for charging in coilguns, or recuperation cycle of coilguns with halfbridge scheme.

Changing the initial conditions in (3a) $i(0) = 0$ to $i(0) = i_0$, we get the following solution (only underdamped circuit is considered):

$$U_c(t) = U_0 e^{-\frac{t}{\tau_L}} \left[\cos\left(\frac{t}{\tau_L} \sqrt{k-1} \right) + \frac{1+km}{\sqrt{k-1}} \sin\left(\frac{t}{\tau_L} \sqrt{k-1} \right) \right]$$

$$i(t) = i_0 e^{-\frac{t}{\tau_L}} \left[\cos\left(\frac{t}{\tau_L} \sqrt{k-1} \right) - \frac{1+m^{-1}}{\sqrt{k-1}} \sin\left(\frac{t}{\tau_L} \sqrt{k-1} \right) \right] \qquad (10)$$

where parameter $m = (\tau \cdot i_0)/(C \cdot U_0)$ characterizes the initial current value in comparison with other parameters of the system. In a flyback converter m is usually much less than 1 as the energy is delivered to the capacitor by small portions. However m may more then 1 in coilguns.

Curves for (10) is depicted in fig. 5 in arbitrary values (positive current here charges the capacitor, as it is stated in the beginning of this section, see fig. 4).

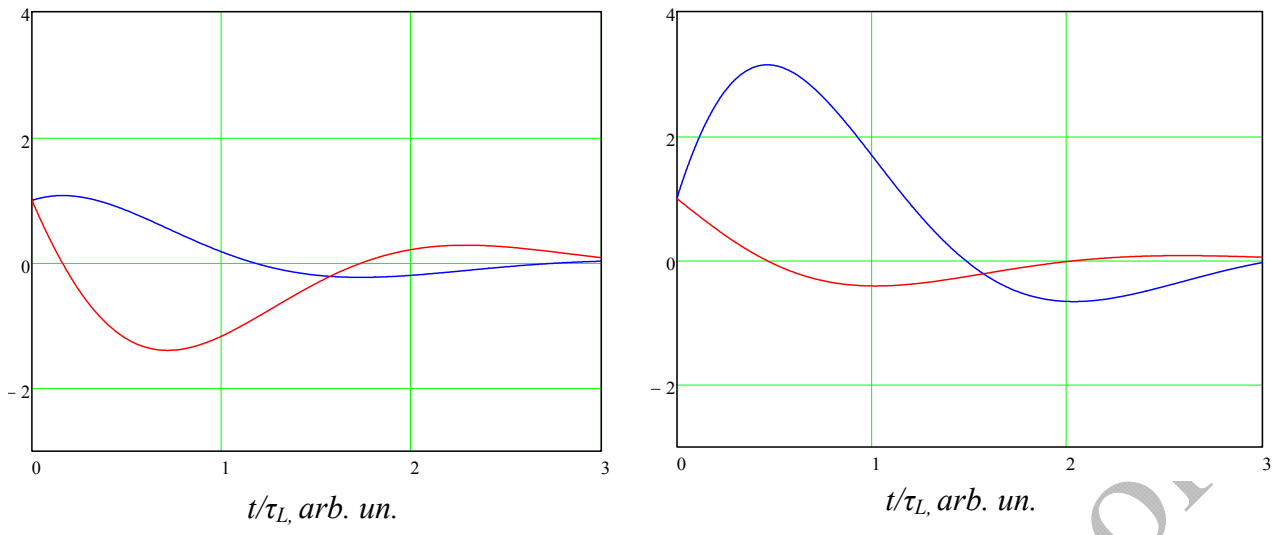


Fig. 5. Arbitrary voltage U_c/U_0 (—) and current i/i_0 (—) for underdamped circuit with k = 5 and m = 0,2 (left) and 2 (right).

One can see that current is continuing charging the capacitor to the voltage more than initial one at first, and then an ordinary periodic discharge is taking place. For small m the maximum voltage to initial voltage ratio is close to 1, but it is more than 3 for m = 2. To finish this chapter, the equation for time moment when the current stops charging and the discharge begins is given:

$$t/\tau_L = \frac{1}{\sqrt{k-1}} arctg\left\{\frac{\sqrt{k-1}}{1+m^{-1}}\right\} \qquad (11)$$

This value describes the duration of the demagnetization cycle of flyback converters and halfbridge coilguns.

5. Expression of the electrical parameters of a circuit through the geometrical parameters of a coil

It is often necessary to establish *RLC*-parameters of a circuit whereas direct measurement is impossible – for example, one doesn't have such testing equipment as *RLC*-meters. It is especially true for inductance and resistance, as the standard capacitors are commonly used in coilguns having the capacitance marked on their cases. In such situation one can try to estimate the needed characteristics basing on *geometrical parameters* of a coil and wire which can be measured by simpler instruments.

In the beginning, let's use famous Wiler formula for inductance:

$$L = 0{,}08 \cdot \frac{D_m{}^2 N^2}{3D_m + 9l + 10t} \qquad (12)$$

where L – inductance, microHenry,
D_m – mean coil diameter, cm,
l – coil length, cm,
t – coil thickness, cm,
N – number of turns (see fig. 6).

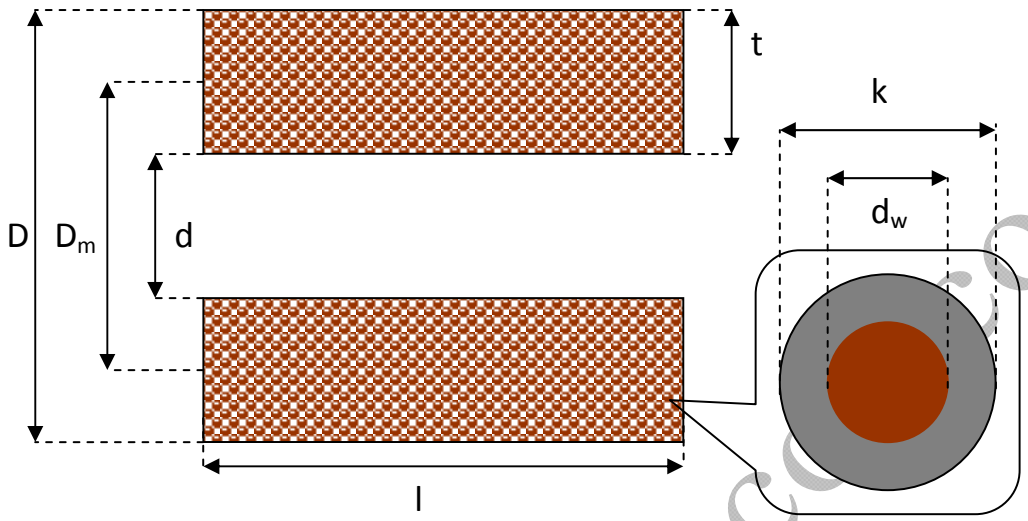


Fig. 6. Designations for eq. (12).

Using the outside coil diameter D and inside one d (it is simply barrel diameter if the coil is wound directly on a barrel of a coilgun) as variables, we easily get

$$L = 0{,}04 \cdot \frac{(D+d)^2 N^2}{13D + 18l - 7d} \qquad (13)$$

N can be simply estimated through the wire diameter k (assuming it is wound toughly, turn to turn):

$$N = \frac{(D-d)l}{2k^2} \qquad (14)$$

So, for inductance we have:

$$L = \frac{0{,}01}{k^4} \cdot \frac{(D+d)^2 (D-d)^2 l^2}{13D + 18l - 7d} \qquad (15)$$

Now what about the resistance ? To calculate it one must multiply the length of the wire by its cross-section $S = \pi d_w^2/4$ and specific resistance ρ (which is $1.75 \cdot 10^{-6}$ Ohm·cm for copper). The length may be estimated via the number of turns N (see (14)) and mean diameter of the single turn D_m . Then the total length of wire in a coil will be

$$l_{\text{пр.}} = \frac{\pi(\mathrm{D}^2 - d^2)l}{4k^2} \qquad (16)$$

Introducing the ratio of the wire diameter without an isolation to the one with isolation $a = d_w/k$, one has finally for the resistance:

$$R = \rho(D^2\text{-}d^2)a^2 l/d_w^{\,4} \qquad (17)$$

Dividing (15) to (17), we have the equation for τ_L:

$$\tau_L = \frac{0{,}02a^2(D^2 - d^2)l}{\rho(13D + 18l - 7d)} \qquad (18)$$

Here τ_L is in microseconds, ρ – in Ohm·cm and all geometrical values – in centimeters.

As one can see, we have an interesting conclusion – the inductive constant of the circuit does not depend on the wire diameter, but only on the coil's geometry.

Now let's fix the inside diameter of the coil d and watch how the value $\omega = \tau_L\rho / a^2 d^2$ is changing in dependence on the length and outside diameter of the coil (see fig. 7).

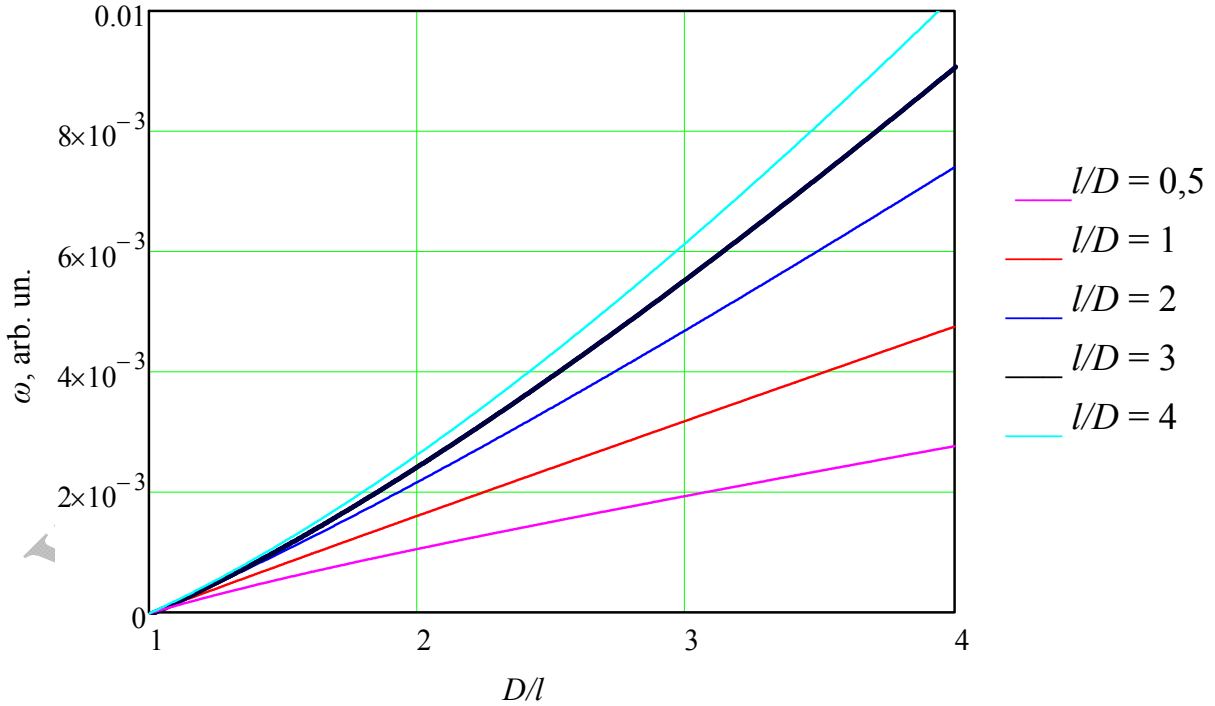


Fig. 7. Specific inductive constant $\omega = \tau_L\rho / a^2 d^2$ in dependence on diameter and length of a coil.

It is clear that τ_L rises monotonically with l and D. I.e. comparing two coils with identical inside diameter, length and resistance, one will find out that the coil with bigger D (the thicker one) will have larger L.

To finish this section we will analyze (with the help of the derived formulae) so called "ideal coil", which has twice inside diameter to length, and $D/d = 3$. The feature of such coil is maximum magnetic field (in its geometrical center) per unit of power in comparison with all coils having the same d and wire resistance. It was shown that the "ideal coil" is not indeed ideal for a coilgun, but many gauss-makers are "in love" with it, so it is interesting to investigate this case.

Substituting $d = 2l$ and $D = 3l$ to (18), we have simple formula for inductive constant of the "ideal coil":

$$\tau_{Lud} = 7{,}8 \cdot 10^{-3} \frac{a^2 l^2}{\rho} \qquad (18a)$$

For instance, copper coil of 2 cm length wound with density of 0.8, has $\tau_L \approx$ 1.72 ms.

Notes.

1) Calculations according to (12)-(18) suggest that the wire lays in concentric layers (they look like plane-parallel layers in axial cross-section of the coil), occupying all volume of the winding. In practice it is not the case, because, for example, the wire can fall down between the turns of the previous layer (see fig. 9). Besides, integer number of turns is never packed within the length of the coil, so winding has defects on its edges. At last, the wire itself can be uneven, interfering the tight fit of the neighboring turns.

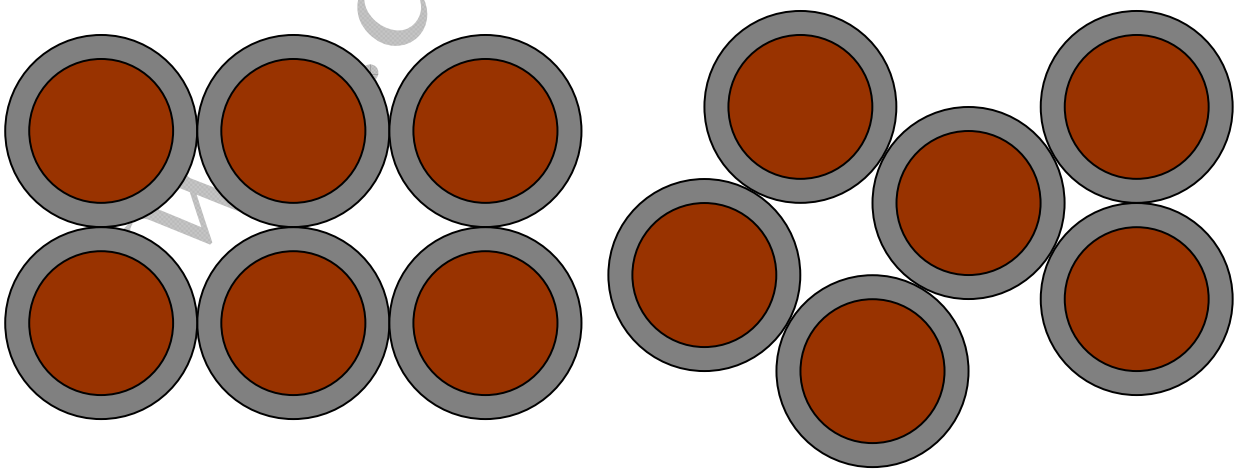


Fig. 9. Ideal winding of plane-parallel layers (left), real winding with defects (right).

Thus, the value of a is empirical and accounts for not only the wire's insulation, but all features above. For the given winding, a may be found by measuring geometrical parameters and resistance of the coil, wound with wire with known diameter of the core d_w. For instance, for copper wire with 1-layer enamel insulation I had a in the range of 0.8….0.85.

2) The formula (12) is analytical approximation for more general equation which can be written as

$$L = f(d,D,l)\cdot N^2 \qquad (19)$$

where $f(d,D,l)$ is form-factor depending only on the geometrical parameters of the coil.

On another hand, the resistance is in direct proportion to the length of wire, and in inverse proportion to its cross-section, which also gives ~ N^2 dependence:

$$R=4\rho\pi D_m N/S \sim \frac{N}{\mathrm{k}^2} \sim \frac{N}{\frac{t\cdot l}{N}} \sim N^2 \qquad (20)$$

(see designations in fig. 6, all parameters not depending on the cross-section are neglected).

Thus, independently on the formula used to calculate the inductance, the ratio L/R is not function of the wire diameter and number of turns, and all conclusions of section 4 concerning τ_L and its connection with the coil's geometry, stay true.